

# SOME NEW DYNAMIC INEQUALITY ON TIME SCALES IN THREE VARIABLES

DEEPAK B. PACHPATTE

ABSTRACT. In this paper we obtain the estimates on some dynamic integral inequalities in three variables which can be used to study certain dynamic equations. We give some applications to convey the importance of our result.

## 1. INTRODUCTION

The study of time scales was initiated in 1989 by Stefan Hilger [5] in his Ph.D dissertation. Since then many authors have studied the dynamic inequalities on time scales. Some analytic inequalities on time scales in one and two variables is studied in [10, 11, 9, 12] by various authors. The authors in [3, 6, 7, 8] have obtained some interesting dynamic integral and iterated inequalities on time scales. Motivated by the results above in this paper we establish new explicit bounds on some dynamic inequalities in three variables which are useful in solving certain dynamic equations.

In what follows  $\mathbb{R}$  denotes the set of real numbers,  $I=[a,b]$  and  $\mathbb{T}$  denotes arbitrary time scales. We say that  $f : \mathbb{T} \rightarrow \mathbb{R}$  is rd-continuous provided  $f$  is continuous right dense point of  $\mathbb{T}$  and has a finite left sided limit at each left dense point of  $\mathbb{T}$  and will be denoted by  $C_{rd}$ . Let  $\mathbb{T}_1$  and  $\mathbb{T}_2$  be two time scales with atleast two points and  $\Omega = \mathbb{T}_1 \times \mathbb{T}_2$  and  $H = \Omega \times I$ . The basic information about time scales can be found in [1, 2]. Now we give the Lemma given in [4] which is required in proving our result.

Lemma [[4]] Let  $u, a, f \in C'_{rd}(\mathbb{T}_1 \times \mathbb{T}_2, \mathbb{R}_+)$  and  $a$  is nondecreasing in each of the variables. If

$$u(x, y) \leq a(x, y) + \int_{x_0}^x \int_{y_0}^y f(s, t) u(s, t) \Delta t \Delta s, \quad (1.1)$$

---

2010 *Mathematics Subject Classification.* 26E70, 34N05, 26D10.

*Key words and phrases.* explicit estimate, integral inequality, dynamic equations, three variables, time scales.

for  $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$  then

$$u(x, y) \leq a(x, y) e^{\int_{y_0}^y f(x, t) \Delta t} (x, x_0), \quad (1.2)$$

for  $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$ .

## 2. MAIN RESULTS

Now we give our main result in the following theorem

**Theorem 2.1** Let  $u, p, q, f \in C_{rd}(H, \mathbb{R}_+)$  and  $c \geq 0$  be a constant. If

$$u(x, y, z) \leq p_1(x, y, z) + p_2(x, y, z) \int_{x_0}^x \int_{y_0}^y \int_a^b f(s, \tau, q) u(s, \tau, q) \Delta q \Delta \tau \Delta s, \quad (2.1)$$

for  $(x, y, z) \in H$ , then

$$u(x, y, z) \leq p_1(x, y, z) + p_2(x, y, z) C(x, y) e_{Q(x, y, z)}(x, x_0), \quad (2.2)$$

where

$$Q(x, y, z) = \int_{y_0}^y \int_a^b f(s, \tau, q) p_2(s, \tau, q) \Delta q \Delta \tau \Delta s, \quad (2.3)$$

$$C(x, y) = \int_{x_0}^x \int_{y_0}^y \int_a^b f(s, \tau, q) p_1(s, \tau, q) \Delta q \Delta \tau \Delta s. \quad (2.4)$$

**Proof.** Now let

$$M(s, t) = \int_a^b f(s, \tau, q) p_2(s, \tau, q) \Delta q. \quad (2.5)$$

Then (2.1) becomes

$$u(x, y, z) \leq p_1(x, y, z) + p_2(x, y, z) \int_{x_0}^x \int_{y_0}^y M(s, \tau) \Delta \tau \Delta s. \quad (2.6)$$

Now put

$$W(x, y) = \int_{x_0}^x \int_{y_0}^y M(s, \tau) \Delta \tau \Delta s. \quad (2.7)$$

Then  $W(x, y_0) = W(x_0, y) = 0$  and

$$u(x, y, z) \leq p_1(x, y, z) + p_2(x, y, z) W(x, y). \quad (2.8)$$

From (2.7), (2.5), (2.8) we have

$$\begin{aligned}
& W^{\Delta_1 \Delta_2}(x, y) \\
&= M(x, y) \\
&= \int_a^b f(x, y, q) u(x, y, q) \Delta q \\
&\leq \int_a^b f(x, y, q) [p_1(x, y, z) + p_2(x, y, z) W(x, y)] \Delta q \\
&= W(x, y) \int_a^b f(x, y, q) u(x, y, q) \Delta q + \int_a^b f(x, y, q) p_1(x, y, q) \Delta q \\
&= \int_a^b f(x, y, q) p_2(x, y, q) \Delta q + \int_a^b f(x, y, q) p_1(x, y, q) \Delta q. \tag{2.9}
\end{aligned}$$

Now from (2.9) above we have by taking delta integral

$$\begin{aligned}
W^{\Delta_1}(x, y) &\leq \int_{y_0}^y \int_a^b W(x, \tau) f(x, \tau, q) p_2(x, \tau, q) \Delta q \Delta \tau \\
&\quad + \int_{y_0}^y \int_a^b f(x, \tau, q) p_1(x, \tau, q) \Delta q \Delta \tau. \tag{2.10}
\end{aligned}$$

Again delta integrating above (2.10) we have

$$\begin{aligned}
W(x, y) &\leq \int_{x_0}^x \int_{y_0}^y \int_a^b W(s, \tau) f(s, \tau, q) p_2(s, \tau, q) \Delta q \Delta \tau \\
&\quad + \int_{x_0}^x \int_{y_0}^y \int_a^b f(s, \tau, q) p_1(s, \tau, q) \Delta q \Delta \tau. \tag{2.11}
\end{aligned}$$

Put

$$B(x, y) = \int_a^b f(x, y, q) p_2(x, y, q) \Delta q,$$

and

$$C(x, y) = \int_{x_0}^x \int_{y_0}^y \int_a^b f(s, \tau, q) p_1(s, \tau, q) \Delta q \Delta \tau.$$

We get from (2.11)

$$W(x, y) \leq \int_{x_0}^x \int_{y_0}^y B(s, \tau) W(s, \tau) \Delta \tau \Delta s + C(x, y). \quad (2.12)$$

Clearly  $C(x, y)$  is nondecreasing in  $\Omega$  then applying Lemma to (2.12), we get

$$W(x, y) \leq C(x, y) e_{\overline{Q}(x, y)}(x, x_0), \quad (2.13)$$

where

$$\overline{Q}(x, y) = \int_{y_0}^y B(x, \tau) \Delta \tau. \quad (2.14)$$

Now using (2.13) in

$$u(x, y, z) \leq p_1(x, y, z) + p_2(x, y, z) W(x, y),$$

we get the result (2.2).

### 3. APPLICATIONS

Now in this section we give some applications of our results. Consider the dynamic integral equation of the form

$$u(x, y, z) = g(h, y, z) + \int_{x_0}^x \int_{y_0}^y \int_a^b F(x, y, z, s, t, q) \Delta q \Delta t \Delta s, \quad (3.1)$$

for  $(x, y, z) \in H$  where  $g \in C_{rd}(H, \mathbb{R})$ ,  $F \in C_{rd}(H^2 \times \mathbb{R}, \mathbb{R})$ .

Now our next theorem deals with the estimate of solution of (3.1).

**Theorem 3.1** Suppose the function  $F$  in (3.1) satisfy the conditions

$$|F(x, y, z, s, t, q, u)| \leq r(x, y, z) f(s, t, q) |u|, \quad (3.2)$$

where  $r, f \in C_{rd}(H, \mathbb{R})$ . If  $u(x, y, z)$  is a solution of equation (3.1) then

$$|u(x, y, z)| \leq |g(h, y, z)| + r(x, y, z) C_2(x, y, z) e_{Q_2(x, y, z)}(x, x_0), \quad (3.3)$$

where

$$C_2(x, y, z) = \int_{x_0}^x \int_{y_0}^y \int_a^b f(s, t, q) |g(s, t, q)| \Delta q \Delta t \Delta s, \quad (3.4)$$

$$Q_2(x, y, z) = \int_{y_0}^y \int_a^b f(x, t, q) r(x, t, q) \Delta q \Delta t, \quad (3.5)$$

for  $(x, y, z) \in H$ .

**Proof.** Let  $u \in C_{rd}(H, \mathbb{R})$  be a solution of (3.1), we have

$$\begin{aligned} |u(x, y, z)| &\leq |g(h, y, z)| + \int_{x_0}^x \int_{y_0}^y \int_a^b |F(x, y, z, s, t, q)| \Delta q \Delta t \Delta s \\ &\leq |g(h, y, z)| \\ &\quad + r(x, y, z) \int_{x_0}^x \int_{y_0}^y \int_a^b f(s, t, q) |u(s, t, q)| \Delta q \Delta t \Delta s. \end{aligned} \quad (3.6)$$

Now applying the Theorem (2.1) gives the estimate (3.3).

Now for obtaining estimates in our next theorem we suppose that  $F$  satisfies Lipschitz type conditions.

**Theorem 3.2** Suppose that the function  $F$  in (3.1) satisfies the condition

$$|F(x, y, z, s, t, q, u) - F(x, y, z, s, t, q, v)| \leq r(x, y, z) f(s, t, q) |u - v|, \quad (3.7)$$

where  $r, f \in C_{rd}(H, \mathbb{R})$ . If  $u(x, y, z)$  is a solution of (3.1) then

$$\begin{aligned} |u(x, y, z) - g(x, y, z)| \\ \leq k(x, y, z) + r(x, y, z) C_3(x, y, z) e_{Q_2(x, y, z)}(x, x_0), \end{aligned} \quad (3.8)$$

for  $(x, y, z) \in H$  where

$$C_3(x, y, z) = \int_{x_0}^x \int_{y_0}^y \int_a^b f(s, t, q) |k(s, t, q)| \Delta q \Delta t \Delta s, \quad (3.9)$$

and

$$k(x, y, z) = \int_{x_0}^x \int_{y_0}^y \int_a^b |F(x, y, z, s, t, q, g(s, t, q))| \Delta q \Delta t \Delta s, \quad (3.10)$$

for  $(x, y, z) \in H$ .

**Proof.** Let  $u \in C_{rd}(H, \mathbb{R})$  be a solution of equation (3.1). Then we have

$$\begin{aligned} |u(x, y, z) - g(x, y, z)| \\ \leq \int_{x_0}^x \int_{y_0}^y \int_a^b |F(x, y, z, s, t, q, u(s, t, q))| \Delta q \Delta t \Delta s \end{aligned}$$

$$\begin{aligned}
&\leq \int_{x_0}^x \int_{y_0}^y \int_a^b |F(x, y, z, s, t, q, u(s, t, q)) \\
&\quad - F(x, y, z, s, t, q, u(s, t, q))| \Delta q \Delta t \Delta s \\
&\quad + \int_{x_0}^x \int_{y_0}^y \int_a^b |F(x, y, z, s, t, q, g(s, t, q))| \Delta q \Delta t \Delta s \\
&\leq k(x, y, z) \\
&\quad + r(x, y, z) \int_{x_0}^x \int_{y_0}^y \int_a^b f(s, t, q) |u(s, t, q) - h(s, t, q)| \Delta q \Delta t \Delta s, \quad (3.11)
\end{aligned}$$

for  $(x, y, z) \in H$ .

Now an application of theorem (2.1) to (3.11) gives the estimate (3.8).

Now we consider equation (3.1) and the integral equation

$$h(x, y, z) = v(x, y, z) + \int_{x_0}^x \int_{y_0}^y \int_a^b G(x, y, z, s, t, q, h(x, y, z)) \Delta q \Delta t \Delta s, \quad (3.12)$$

for  $v \in C_{rd}(H, \mathbb{R}), G \in C_{rd}(H^2 \times \mathbb{R}, \mathbb{R})$ .

Now we give the following theorem.

**Theorem 3.3** Suppose the function  $F$  in (3.1) satisfies the condition (3.7) then for every solution  $h \in C_{rd}(H, \mathbb{R})$  of (3.11) and  $u \in C_{rd}(H, \mathbb{R})$  solution of (3.1) we have the estimates

$$\begin{aligned}
|u(x, y, z) - h(x, y, z)| &\leq [\bar{g}(x, y, z) + \bar{k}(x, y, z)] \\
&\quad + r(x, y, z) C_4(x, y, z) e_{Q_2(x, y, z)}(x, x_0). \quad (3.13)
\end{aligned}$$

for  $(x, y, z) \in H$  in which

$$C_4(x, y, z) = \int_{x_0}^x \int_{y_0}^y \int_a^b f(s, t, q) [\bar{g}(s, t, q) + \bar{k}(s, t, q)] \Delta q \Delta t \Delta s. \quad (3.14)$$

$$\bar{g}(x, y, z) = |g(x, y, z) - v(x, y, z)|. \quad (3.15)$$

$$\begin{aligned}
\bar{k}(x, y, z) &= \int_{x_0}^x \int_{y_0}^y \int_a^b |F(x, y, z, s, t, q, h(s, t, q)) \\
&\quad - G(x, y, z, s, t, q, h(s, t, q))| \Delta q \Delta t \Delta s. \quad (3.16)
\end{aligned}$$

for  $(x, y, z) \in H$ .

**Proof.** Since  $u(x, y, z)$  and  $v(x, y, z)$  are respectively solutions of (3.1) and (3.12) we have

$$\begin{aligned}
& |u(x, y, z) - h(x, y, z)| \\
& \leq [\bar{g}(x, y, z) + \bar{k}(x, y, z)] \\
& + \int_{x_0}^x \int_{y_0}^y \int_a^b |F(x, y, z, s, t, q, u(s, t, q)) \\
& - F(x, y, z, s, t, q, h(s, t, q))| \Delta q \Delta t \Delta s \\
& + \int_{x_0}^x \int_{y_0}^y \int_a^b |F(x, y, z, s, t, q, u(s, t, q)) \\
& - G(x, y, z, s, t, q, h(s, t, q))| \Delta q \Delta t \Delta s \\
& \leq \bar{g}(x, y, z) + \bar{k}(x, y, z) \\
& + r(x, y, z) \int_{x_0}^x \int_{y_0}^y \int_a^b f(s, t, q) |u(s, t, q) - h(s, t, q)| \Delta q \Delta t \Delta s. \quad (3.17)
\end{aligned}$$

Now an application of Theorem 2.1 to (3.17) yields (3.13).

## REFERENCES

- [1] M. Bohner and A. Peterson, Dynamic equations on time scales, *Birkhauser Boston/Berlin*, (2001).
- [2] M. Bohner and A. Peterson, Advances in Dynamic equations on time scales, *Birkhauser Boston/Berlin*, (2003).
- [3] E.A. Bohner, M. Bohner and F. Akin, Pachpatte inequalities on on time scales, *J. Inequal. Pure Appl. Math.*, 6(1)(2005), Art. 6.
- [4] R.A.C. Ferreira, D.F.M. Torres, Some linear and nonlinear integral inequalities on tiem scales in two independent variables, *Nonlinear Dynamics and systems Theory*, Vol. 9, no 2, pp. 161-169, 2009.
- [5] S. Hilger, Analysis on Measure chain-A unified approach to continuous and discrete calculus, *Results. Math.*, 18:18-56, 1990.
- [6] D. B. Pachpatte, Explicit estimates on integral inequalities with time scale, *J. Inequal. Pure. Appl. Math.*, Vol. 7, Issue 4, Artivle 143, 2006.
- [7] D. B. Pachpatte, Integral Inequalits for partial dynamic equations on time scales, *Electron. J. Differential Equations*, Vol. 2012 (2012), No. 50, 1-7.
- [8] D. B. Pachpatte, Estimates of Certain Iterated dynamic inequalities on time scales *Qual. Theory Dyn. Syst.* Vol 13, No. 2, 2014.
- [9] Y. Suna, T. Hassanb, Some nonlinear dynamic integral inequalities on time scales, *Appl. Math. Comput.*, Vol 220, 2013, P. 221-225.

- [10] S. Hussain, M. A. Latif, M. Alomari, Generalized double-integral Ostrowski type inequalities on time scales, *Appl. Math. Lett.*, Vol 24, Issue 8, Aug 2011, P. 1461-1467.
- [11] U. M. Ozkan, M. Z. Sarikaya, H. Yildirim Extensions of certain integral inequalities on time scales, *Appl. Math. Lett.*, Vol. 21, Issue 10, Oct. 2008, P. 993-1000.
- [12] C. Yeh Ostrowski inequality on time scales, *Appl. Math. Lett.*, Vol. 21, Issue 4, Apr 2008, P. 404-409.

DEEPAK B. PACHPATTE

DEPARTMENT OF MATHEMATICS, DR. BABASAHEB AMBEDKAR MARATHWADA  
UNIVERSITY, AURANGABAD, MAHARASHTRA 431004, INDIA

*E-mail address:* pachpatte@gmail.com